On Changes in Frobenius Numbers

Abstract

Given a set $A = \{a_1, \ldots, a_k : \ a_i \in \mathbb{Z}_{\geq 0}\}$ of coprime positive integers, define $\Gamma(A) := \{a_1x_1 + \ldots a_kx_k : x_i \in \mathbb{Z}_{\geq 0}\}$. As $\Gamma^c(A) := \mathbb{N} \setminus \Gamma(A)$ is a finite set, we define

$$\mathtt{g}(A) := \max \Gamma^c(A), \ \ \mathtt{n}(A) := |\Gamma^c(A)|, \ \ \text{the sum} \ \ s(A) := \sum_{n \in \Gamma^c(A)} n.$$

Formulae for these functions are well known for the case $k = 2$. Let a 2-set be $\{a, b\}$, then we know

$$\mathtt{g}(a,b) = (a-1)(b-1)-1, \ \ \mathtt{n}(a,b) = \frac{1}{2}(a-1)(b-1), \ \ s(a,b) = \frac{1}{12}(a-1)(b-1)(2ab-a-b-1)$$

Using elementary tools, we characterize c for which $\mathtt{g}(a,b) - \mathtt{g}(a,b,c)$ belongs to the set $\{a, b, a+b, 2a, 2b, 2a+b, a+2b, 3a, 3b\}$ and in more generally, when the difference belongs to $\{ka, kb, ka+b, a+kb\}$ with some restrictions for k in case when diffrence is $ka+b$.

We also determine $\mathtt{g}(A), \mathtt{n}(A)$, $s(A)$ for certain families $A = \{a, b, c\}$. Along with this, for the same families A, we find the set $S^{\star}(A) = \{n \in \Gamma^c(A) : n + \Gamma(A)^{\star} \subset \Gamma(A)^{\star}\}$. Algorithmically, for general set $\{a, b, c\}$ we establish the difference $\mathbf{m}_{bi} - \mathbf{m}^{\star}_{bi}$ for each $i \in \{0, 1, \ldots, a\}$, where $\mathbf{m}_{bi}$ and $\mathbf{m}^{\star}_{bi}$ denote smallest integer congruent to bi mod a in $\Gamma(\{a,b\})$ and $\Gamma(\{a,b,c\})$, respectively. This enables us in giving a very neat formula for $\mathtt{n}(a,b,c), \mathtt{g}(a,b,c), s(a,b,c)$.