On Changes in Frobenius Numbers

Abstract

Given a set \( A = \{a_1, \ldots, a_k : a_i \in \mathbb{Z}_{\geq 0}\} \) of coprime positive integers, define \( \Gamma(A) := \{a_1x_1 + \ldots + a_kx_k : x_i \in \mathbb{Z}_{\geq 0}\} \). As \( \Gamma^c(A) := \mathbb{N} \setminus \Gamma(A) \) is a finite set, we define

\[
\begin{align*}
g(A) &:= \max \Gamma^c(A), \\
n(A) &:= |\Gamma^c(A)|, \\
s(A) &:= \sum_{n \in \Gamma^c(A)} n.
\end{align*}
\]

Formulae for these functions are well known for the case \( k = 2 \). Let a 2-set be \( \{a, b\} \), then we know

\[
\begin{align*}
g(a, b) &= (a-1)(b-1)-1, \\
n(a, b) &= \frac{1}{2}(a-1)(b-1), \\
s(a, b) &= \frac{1}{12}(a-1)(b-1)(2ab-a-b-1)
\end{align*}
\]

Using elementary tools, we characterize \( c \) for which \( g(a, b) - g(a, b, c) \) belongs to the set \( \{a, b, a+b, 2a, 2b, 2a+b, a+2b, 3a, 3b\} \) and in more generally, when the difference belongs to \( \{ka, kb, ka+b, a+kb\} \) with some restrictions for \( k \) in case when difference is \( ka+b \).

We also determine \( g(A), n(A), s(A) \) for certain families \( A = \{a, b, c\} \). Along with this, for the same families \( A \), we find the set \( S^*(A) = \{n \in \Gamma^c(A) : n + \Gamma(A)^* \subset \Gamma(A)^*\} \). Algorithmically, for general set \( \{a, b, c\} \) we establish the difference \( m_{bi} - m_{bi}^* \) for each \( i \in \{0, 1, \ldots, a\} \), where \( m_{bi} \) and \( m_{bi}^* \) denote smallest integer congruent to \( bi \mod a \) in \( \Gamma(\{a, b\}) \) and \( \Gamma(\{a, b, c\}) \), respectively. This enables us in giving a very neat formula for \( n(a, b, c), g(a, b, c), s(a, b, c) \).