Graph coloring and its variations: Structural and algorithmic study

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ABSTRACT

With its origin embedded in attempts to solve the famous Four Color Problem, graph coloring has flourished into one of the most important and well studied areas of current research. Graph coloring, in general, is an assignment of colors, integers from $\{1, 2, \ldots, k\}$, to the vertex set or edge set of a graph. Various variations of graph coloring have been defined and studied in literature, owing to their practical significance. In this thesis, we study the structural and algorithmic aspects of four variations of graph coloring, namely, (i) neighbor-sum-2-distinguishing edge-weighting, (ii) injective coloring, (iii) exact square coloring, and (iv) injective edge coloring.

A *neighbor-sum-distinguishing W-edge-weighting* of a graph $G = (V, E)$ is an assignment ω of weights from a set of integers W to the set of edges E of G such that for every pair of adjacent vertices, the incident sums induced by the edge-weighting are different, where the incident sum of a vertex v induced by the edge-weighting ω is $\sigma(v) = \sum_{u \in N(v)} \omega(uv)$, where $N(v)$ is the set of neighbors of v in G. Whereas, a *neighbor-sum-2-distinguishing W-edge-weighting* of G is a neighbor-sum-distinguishing W-edge-weighting such that the incident sums of every pair of adjacent vertices differ by at least 2. A recently proposed conjecture about neighbor-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting states that any graph with no component isomorphic to K_2 admits a neighbor-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting. In this thesis, we prove that deciding whether there exists a neighbor-sum-2-distinguishing $\{1, 3\}$-edge-weighting is NP-complete for bipartite graphs. We present an algorithm that computes a neighbor-sum-2-distinguishing $\{1, 3\}$-edge-weighting of the central graph of any graph in polynomial time and thus prove the above conjecture for the central graph of any graph. We study another aspect of neighbor-sum-2-distinguishing edge-weighting of a graph G, namely, the minimum number of incident sums used by a neighbor-sum-2-distinguishing $\{1, 3\}$-edge-weighting, which is denoted by $\gamma_{\Sigma>1,\{1,3\}}(G)$. We prove that the problems of deciding whether there exists a neighbor-sum-2-distinguishing $\{1, 3\}$-edge-weighting of G from a given set of sums, and deciding whether $\gamma_{\Sigma>1,\{1,3\}}(G) \leq k$ are NP-complete for bipartite graphs. However, we provide both upper and lower bounds on $\gamma_{\Sigma>1,\{1,3\}}(C(G))$ for central graphs of bipartite graphs and split graphs. We also propose an algorithm that computes the optimal neighbor-sum-2-distinguishing $\{1, 3\}$-edge-weighting of the central graphs of cycles and paths in polynomial time.

A vertex coloring of a graph $G = (V, E)$ that uses k colors is called an *injective k-coloring* of G if no two vertices having a common neighbor have the same color. The minimum k for which G has an *injective k-coloring* is called the *injective chromatic number* of G. Given a graph G and a positive integer k, the DECIDE INJECTIVE COLORING PROBLEM is to decide whether G admits an injective k-coloring. It is known that DECIDE INJECTIVE COLORING PROBLEM is NP-complete for bipartite graphs. In this thesis, we strengthen this result by proving that this problem remains NP-complete for perfect elimination bipartite graphs, star-convex bipartite graphs and comb-convex bipartite graphs, which are proper subclasses of bipartite graphs. Moreover, we show that for every $\epsilon > 0$, it is not possible to efficiently approximate the injective chromatic number of a perfect elimination bipartite graph within a factor of $n^{\frac{1}{3}-\epsilon}$ unless ZPP = NP. However, we propose a linear time algorithm for biconvex bipartite graphs and $O(nm)$ time algorithm for convex bipartite graphs for finding the optimal injective coloring. We prove that the injective chromatic number of a chordal bipartite graph can be determined in polynomial

time. It is known that DECIDE INJECTIVE COLORING PROBLEM is NP-complete for chordal graphs. We give linear time algorithms for computing the injective chromatic number of proper interval graphs and threshold graphs, which are proper subclasses of chordal graphs. DECIDE INJECTIVE COLORING PROBLEM is also known to be NP-complete for split graphs. We show that DECIDE INJECTIVE COLORING PROBLEM remains NP-complete for $K_{1,t}$-free split graphs for $t \geq 4$ and polynomially solvable for $t \leq 3$.

A vertex coloring of a graph $G = (V, E)$ is called an *exact square coloring* of G if any two vertices at distance exactly 2 receive different colors. The minimum number of colors required by an exact square coloring is called the *exact square chromatic number* of G and is denoted by $\chi^{[\#2]}(G)$. Given a graph G and a positive integer k, the EXACT SQUARE COLORING PROBLEM is to decide whether G admits an exact square coloring using k colors. It is known that EXACT SQUARE COLORING PROBLEM is NP-complete for chordal graphs. In this thesis, we strengthen this result by proving that this problem remains NP-complete for undirected path graphs, which is a proper subclass of chordal graphs. However, we give linear time algorithms for computing the exact square chromatic number of proper interval graphs and threshold graphs. Moreover, for a proper interval graph G, we show that $\chi^{[\#2]}(G) \leq 3$. We also propose a polynomial time algorithm to produce an exact square coloring of a block graph G using at most $\chi^{[\#2]}(G) + 1$ colors. Next, we study a lower bound of $\chi^{[\#2]}(G)$. A subset S of vertices of a graph $G = (V, E)$ is called an *exact square clique* of G if the distance between any two vertices in S is exactly 2. The cardinality of the maximum exact square clique of G is called the exact square clique number of G and is denoted by $\omega^{[\#2]}(G)$. Clearly, $\omega^{[\#2]}(G) \leq \chi^{[\#2]}(G)$. Given a graph G and a positive integer k, the problem of deciding whether $\omega^{[\#2]}(G)$ is at least k, is known to be NP-complete for bipartite graphs and chordal graphs. In this thesis, we strengthen these results by proving that this problem remains NP-complete for undirected path graphs, perfect elimination bipartite graphs, star-convex bipartite graphs and comb-convex bipartite graphs. We also compute the exact value of $\omega^{[\#2]}(G)$ for proper interval graphs, threshold graphs, block graphs and convex bipartite graphs.

An *injective k-edge-coloring* of a graph $G = (V, E)$ is an assignment $\omega : E \rightarrow \{1, 2, \ldots, k\}$ of colors to the edges of G such that any two edges e and f receive distinct colors if there exists an edge $g = xy$ different from e and f such that e is incident on x and f is incident on y. The minimum value of k for which G admits an injective k-edge-coloring is called the *injective chromatic index* of G and is denoted by $\chi'_i(G)$. Clearly, an injective edge coloring is the natural edge-version of the notion of an *injective coloring*. Given a graph G and a positive integer k, the INJECTIVE EDGE COLORING PROBLEM is to decide whether G admits an injective k-edge-coloring. It is known that INJECTIVE EDGE COLORING PROBLEM is NP-complete for general graphs. In this thesis, we strengthen this result by proving that INJECTIVE EDGE COLORING PROBLEM is NP-complete for bipartite graphs by proving that this problem remains NP-complete for perfect elimination bipartite graphs and star-convex bipartite graphs. However, we propose linear time algorithms for computing the injective chromatic index of trees and chain graphs, which is a proper subclass of both perfect elimination bipartite graphs and and star-convex bipartite graphs. We also propose linear time algorithm for computing the injective chromatic index of threshold graphs. A graph G is known as a *perfect EIC-graph* if $\chi'_i(G)$ is equal to the number of edges in a maximum clique of G. We prove that threshold graphs are perfect EIC-graphs.

Various operations defined over graphs form an important method to both construct new graphs and structurally characterize particular graph classes. In this thesis, we have also studied these variations of graph coloring under some standard graph operations and products, including Cartesian product, strong product, lexicographic product, corona product, edge corona product, join, subdivision and Mycielskian of a graph.