Abstract

Let K be a field of characteristic zero and $K[X] = K[x_1, x_2, \ldots, x_n]$ be the polynomial algebra in n variables over K. The main emphasis of this thesis is to study the kernel and the image of K-derivations and $K\mathcal{E}$-derivations defined over $K[X]$. As the kernel and the image are fundamental concepts attached to any morphism on an algebraic structure, it is desirable to study them. For a K-derivation or $K\mathcal{E}$-derivation d of $K[X]$, it is easy to see that kernel $K[X]^d$ is a K-subalgebra of $K[X]$. Indeed, for a $K\mathcal{E}$-derivation δ, $K[X]^\delta$ coincides with the K-subalgebra of elements of $K[X]$ that are fixed by the corresponding endomorphism. The study of the kernel of a derivation is closely related to the well-known Cancellation Problem and the Fourteenth Problem of Hilbert.

If d is a K-derivation of $K[X]$, then it uniquely extends to a K-derivation of $K(X)$, which one continues to denote by d. The kernel $K[X]^d$ is sometimes referred to as the algebra of constants of d, and $K(X)^d$ as the field of rational constants of d. The structure of the field of rational constants of K-derivation is closely connected with the existence of Darboux polynomials. It is evident that if the K-derivation is without Darboux polynomials, then the field of rational constants of the K-derivation is trivial. However, the reverse conclusion is generally not true.

This thesis is divided into two parts. The first part of this thesis focuses on studying the rational constants and Darboux polynomials of a generalized cyclotomic derivation of $K[X]$. Further, it is shown that the generalised cyclotomic derivation is without Darboux polynomials if and only if the field of rational constants of the derivation is trivial. Furthermore, the result is also studied in the tensor product of polynomial algebras.

As compared to the kernel of a K-derivation and $K\mathcal{E}$-derivation, a little is known the about the image of a K-derivation and $K\mathcal{E}$-derivation. The structural properties of the image of a K-derivation (or $K\mathcal{E}$-derivation), significantly differ from those of the kernel of a K-derivation. The image of derivations, unlike its kernel, doesn't necessarily exhibit closure under multiplication. However, the image of certain K-derivations and $K\mathcal{E}$-derivations possess an algebraic structure called Mathieu-Zhao subspace. Zhao introduced the notion of Mathieu-Zhao subspace in connection to the Mathieu conjecture and the Image conjecture. Notably, both the conjectures mentioned above are motivated by and imply the well-known Jacobian conjecture proposed by Keller in 1939. Although this notion of Mathieu-Zhao subspace is mainly motivated by the study of these conjectures, this new idea gives a natural but highly non-trivial generalization to the fundamental concept of ideals in a ring makes it more intriguing. One can easily observe that every ideal is a Mathieu-Zhao subspace but not vice versa.

Zhao in 2018 proposed two conjectures related to the image of a locally finite K-derivation (or $K\mathcal{E}$-derivation) and the image of ideals under a locally nilpotent K-derivation (or $K\mathcal{E}$-derivation). The conjecture on the image of a locally finite derivation K-derivation (or $K\mathcal{E}$-derivation) is known as LFED conjecture, which asserts that if A is a K-algebra and d is a locally finite K-derivation (or $K\mathcal{E}$-derivation) of A, then the image of d is a Mathieu-Zhao subspace of A. Further, the conjecture proposed on the image of an ideal under a locally nilpotent K-derivation (or $K\mathcal{E}$-derivation) is known as the LNED conjecture. It states that if A is a K-algebra and d is a locally nilpotent K-derivation (or $K\mathcal{E}$-derivation) of A, then d maps every ideal of A to a Mathieu-Zhao subspace of A. Studying the above mentioned conjectures is warranted not only because they are related to the Jacobian conjecture but also because they might give new insight into locally finite and locally nilpotent K-derivations (or $K\mathcal{E}$-derivations).

The second part of the thesis focuses on the study of the image of linear K-derivations and $K\mathcal{E}$-derivations of $K[X]$. Note that every linear K-derivation and $K\mathcal{E}$-derivation is locally finite. We prove that if the eigenvalues of the associated matrix of a linear K-derivation d are linearly independent over $\mathbb{N}_0$, then the image of d is a maximal ideal in $K[X]$. Further, it is shown that the LFED conjecture holds for some linear K-derivations of $K[x_1, x_2, x_3, x_4]$. Furthermore, we assume certain conditions on the eigenvalues corresponding to the linear K-endomorphism $\mathrm{I} - \delta$ of $K[x_1, x_2, x_3, x_4]$, where δ is a linear $K\mathcal{E}$-derivation of $K[x_1, x_2, x_3, x_4]$ and prove that the LFED conjecture is valid for linear $K\mathcal{E}$-derivations of $K[x_1, x_2, x_3, x_4]$.

Moreover, we prove that for a linear K-derivation d of $K[X]$ and the maximal ideal $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$ of $K[X]$, if $d(\mathfrak{m})$ is a Mathieu-Zhao subspace of $K[X]$, then the image of every $\mathfrak{m}$-primary ideal under d forms a Mathieu-Zhao subspace of $K[X]$. Additionally, we observe that the image of all monomial ideals under the K-derivation $d = f\partial_{x_1}$ of $K[X]$, for $f \in K[X]$ forms an ideal of $K[X]$. Finally, we prove that the image of certain monomial ideals under a linear locally nilpotent K-derivation of $K[x_1, x_2, x_3]$ defined by $d = x_2\partial_{x_1} + x_3\partial_{x_2}$ forms a Mathieu-Zhao subspace.